Second Memoir on a New Theory of Symmetric Functions.

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In my first memoir on this subject (Vol. XI, No. 1) I introduced the notion of the "separation" of a partition, but restricted myself to the discussion of rational integral symmetric functions.

In the present memoir I am engaged with functions which are not necessarily integral, but require partitions, with positive, zero, and negative parts for their symbolical expression.

The chief results which I obtain are

(i). A simple proof of a generalized Vandermonde-Waring power law which presents itself in the guise of an invariantive property of a transcendental transformation.

(ii). The law of "Groups of Separations."

(iii). The fundamental law of algebraic reciprocity; the proof here given being purely arithmetical.

(iv). The fundamental law of algebraic expressibility which asserts that certain indicated symmetric functions can be exhibited as linear functions of the separations of any given partition.

(v). The existence is established of a pair of symmetrical tables in association with every partition into positive, zero, and negative parts, of every number positive, zero, or negative.

The results (iv) and (v) are immediate deductions from (iii), which I believe to be a theorem of great importance and a natural origin of research in symmetrical algebra.

Attention may be drawn to the free introduction of the zero part into the partitions; this forms a connecting link between arithmetic and algebra, and

enables us to pass in a novel and natural manner from theorems of quantity to theorems of number. An illustration of this may be found at the conclusion of this memoir, where I have given symmetrical tables of binomial coefficients. By employing zero parts, any algebraic function of one quantity may be expressed by means of partitions, and further, every unsymmetrical algebraic function of the quantity x is expressible as a symmetric function of any arbitrary quantities x in number; this is in fact equivalent to the development of $\phi(x)$, a given rational and integral algebraic function of x, in a series of factorials, but it is interesting as showing that all algebra is in reality included in the algebra of symmetric functions; for this reason I think the theorems here given are entitled to rank as theorems in general algebra, and should not be regarded as appertaining exclusively to symmetrical algebra.

In one or two succeeding memoirs I hope to be permitted to further develop the theory of the $X - x$ transformation which possesses many properties of great elegance, and to exhibit, with some approach to completeness, the theory of the allied differential operations, a large and important part of the subject upon which I have not entered in these two memoirs, although I have it by me in manuscript.

Readers should consult "Symmetric Functions and the Theory of Distributions," Lond. Math. Soc., Vol. XIX, p. 220, and "Théorie des Formes Binaires," by Faà de Bruno.

SECTION 1.

1. The theory of symmetric functions is a part of the general theory of permutations, combinations and distributions. Formulae in the former are merely elegant analytical expressions of propositions in the latter theory; this fact I have dwelt upon at some length in a paper, "Symmetric Functions and the Theory of Distributions," Proceedings of the London Mathematical Society, Vol. XIX, p. 220 et seq.

As an illustration, I give the interpretations of two well known theorems in symmetric functions and refer readers to the paper above quoted for the necessary explanations and elucidations.

2. If

$$(1 - a_1x + a_2x^2 - a_3x^3 + \dots)^{-1} = 1 + h_1x + h_2x^2 + h_3x^3 + \dots:,$$

then a_m and h_m are designated respectively "the elementary symmetric function of weight m," and "the homogeneous product sum of weight m" of the quantities

$$\alpha, \beta, \gamma, \delta, \ldots,$$

where

$$1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)\ldots.$$

We have the well known theorems

$$(-)^m a_m = \sum \frac{(-)^{\Sigma\lambda}(\Sigma\lambda)!}{\lambda_1!\,\lambda_2!\,\lambda_3!\ldots} h_1^{\lambda_1} h_2^{\lambda_2} h_3^{\lambda_3}\ldots, \qquad \text{(i)}$$

the summation being controlled by the relation $\Sigma s\lambda_s = m$ and

$$(-)^m h_m = \sum \frac{(-)^{\Sigma\lambda}(\Sigma\lambda)!}{\lambda_1!\,\lambda_2!\,\lambda_3!\ldots} a_1^{\lambda_1} a_2^{\lambda_2} a_3^{\lambda_3}\ldots, \qquad \text{(ii)}$$

with, as before, the relation $\Sigma s\lambda_s = m$.

3. It will be observed that (ii) is derivable from (i) by the interchange of a and h.

4. These formulae give rise respectively to

Theorem I. "Considering n objects of any species whatever, the number of distinct ways of distributing them into an even number of different parcels is precisely equal to the number of distributions into an uneven number of different parcels, except when the objects are all of different species; in this case, the former number is in excess or in defect of the latter number by unity, according as the number of objects is even or uneven."

5. *Theorem* II. "Considering n objects of any species whatever with the restriction that no parcel may contain two objects of the same species, the number of distributions into an even number of different parcels is in excess or in defect by unity of the number of distributions into an uneven number of different parcels according as n, the number of objects, is even or uneven."

6. In these theorems it is to be understood that the phrase "of any species whatever" means that the objects are not restricted to be all of the same kind or to be all of different kinds, but may be of any kinds whatever; the phrase "different parcels" means that no two parcels are of the same description.

7. As an example of the first theorem, suppose there are four objects, say three pears and an apple, we have the distributions:

	Four parcels.	Three parcels.	Two parcels.	One parcel.
	p, p, p, a	pp, p, a	pp, pa	$pppa$
	p, p, a, p	pp, a, p	pa, pp	
	p, a, p, p	p, pp, a	ppp, a	
	a, p, p, p	a, pp, p	a, ppp	
		p, a, pp	ppa, p	
		a, p, pp	p, ppa	
		pa, p, p		
		p, pa, p		
		p, p, pa		
No. =	4	9	6	1

and
$$4+6=9+1,$$
as stated by the theorem.

8. Again take three different objects, say a pear, an apple, and an orange; the distributions are

	Three parcels.	Two parcels.	One parcel.
	p, a, o	pa, o	pao
	p, o, a	o, pa	
	a, p, o	ao, p	
	a, o, p	p, ao	
	o, p, a	op, a	
	o, a, p	a, op	
No. =	6	6	1

and
$$6+1-6=1,$$
as stated by the theorem.

9. As an example of the second theorem, take two pears and two apples, and remember that now no two similar objects can appear in the same parcel; we have thus

	Four parcels.	Three parcels.	Two parcels.	One parcel.
	p, p, a, a	pa, p, a	pa, pa	no way.
	p, a, p, a	pa, a, p		
	a, p, p, a	p, pa, a		
	p, a, a, p	a, pa, p		
	a, p, a, p	p, a, pa		
	a, a, p, p	a, p, pa		
No. =	6	6	1	0

and
$$6+1-(6+0)=1,$$
as should be the case.

SECTION 2.

The Vandermonde-Waring Law.

10. Referring readers to the "Definitions" given on page 2 of my former memoir, I pass on to a further consideration of the separation theorem given on page 19 (loc. cit.), viz.

$$(-)^{l+m+\dots}\frac{(l+m+\dots-1)!}{l!\,m!\dots}S(\lambda^l\mu^m\dots)$$
$$=\sum(-)^{j_1+j_2+\dots}\frac{(j_1+j_2+\dots-1)!}{j_1!\,j_2!\dots}(J_1)^{j_1}(J_2)^{j_2}\dots,$$

where $S(\lambda^l\mu^m\dots)$ denotes the sum of the n^{th} powers of the quantities expressed by means of separations of the partition $(\lambda^l\mu^m\dots)$ of the number n; $(J_1)^{j_1}(J_2)^{j_2}\dots$ is any one of these separations and the summation is in regard to all the separations.

11. I established this theorem in the Proceedings of the London Mathematical Society, Vol. XIX, p. 247 et seq., but having recently obtained a far simpler proof, I give it here as a preparation for a far more general result which will be established subsequently.

12. Write

$$X_1=(1)\,x_1,$$
$$X_2=(2)\,x_2+(1^2)\,x_1^2,$$
$$X_3=(3)\,x_3+(21)\,x_2x_1+(1^3)\,x_1^3,$$
$$X_4=(4)\,x_4+(31)\,x_3x_1+(2^2)\,x_2^2+(21^2)\,x_2x_1^2+(1^4)\,x_1^4,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$X_m=\Sigma\,(m_1m_2m_3\dots)\,x_{m_1}x_{m_2}x_{m_3}\dots,$$

the summation having reference to every partition $(m_1m_2m_3\dots)$ of the number m.

13. We may regard the quantities X_1, X_2, X_3, ... as transformed into the quantities x_1, x_2, x_3, ... by means of these relations, and we may enquire whether there exists a system of invariants of this transformation; whether in fact we can form a system of relations between X_1, X_2, X_3, ... which, to symmetric function multipliers *près*, are equal to the like functions of x_1, x_2, x_3; .... A complete system of such invariants does exist, and they are of fundamental importance.

In the first place, X_1 is such an invariant; the complete system is found in the following manner:

14. I suppose that the symmetric functions on the dexter of the above relations refer to quantities

$$\alpha, \beta, \gamma, \ldots,$$

which I further consider to be infinite in number.

15. I observe that the expression

$$1 + X_1 + X_2 + X_3 + \ldots$$

may be broken up into factors of the form

$$1 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \ldots,$$

so that there is the identity

$$1 + X_1 + X_2 + X_3 + \ldots = \Pi_\alpha (1 + \alpha x_1 + \alpha^2 x_2 + \alpha^3 x_3 + \ldots),$$

a factor appearing for each of the quantities

$$\alpha, \beta, \gamma, \ldots;$$

this relation indeed, from another point of view, serves to define the quantities $X_1, X_2, X_3, \ldots$ in a concise manner and *a posteriori* one is directly convinced of its truth.

16. It is convenient to introduce an arbitrary quantity μ and to write

$$1 + \mu X_1 + \mu^2 X_2 + \mu^3 X_3 + \ldots = \Pi_\alpha (1 + \mu\alpha x_1 + \mu^2\alpha^2 x_2 + \mu^3\alpha^3 x_3 + \ldots)$$

Taking logarithms, we find

$$\log(1 + \mu X_1 + \mu^2 X_2 + \mu^3 X_3 + \ldots) = \sum_\alpha \log(1 + \mu\alpha x_1 + \mu_2\alpha^2 x_2 + \mu^3\alpha^3 x_3 + \ldots);$$

the left-hand side of this identity is, when expanded,

$$\mu X_1 + \mu^2\left(X_2 - \frac{1}{2}X_1^2\right) + \mu^3\left(X_3 - X_2X_1 + \frac{1}{3}X_1^3\right) + \ldots,$$

the general term being

$$\mu^l \sum (-)^{l_1 + l_2 + \ldots + 1} \frac{(l_1 + l_2 + \ldots - 1)!}{l_1!\, l_2! \ldots} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} \ldots,$$

where $$l = \sum l\lambda;$$

whereas the right-hand side has the general term

$$\left\{\sum \alpha^l\right\}\mu^l \sum (-)^{l_1+l_2+\ldots+1} \frac{(l_1+l_2+\ldots-1)!}{l_1!\,l_2!\ldots} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

17. Hence, equating coefficients of like powers of μ, we have a system of invariants shown by the relations

$$X_1 = (1)\,x_1,$$

$$X_2 - \frac{X_1^2}{2} = (2)\left\{x_2 - \frac{x_1^2}{2}\right\},$$

$$X_3 - X_2X_1 + \frac{X_1^3}{3} = (3)\left\{x_3 - x_2x_1 + \frac{x_1^3}{3}\right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum (-)^{l_1+l_2+\ldots-1} \frac{(l_1+l_2+\ldots-1)!}{l_1!\,l_2!\ldots} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2}\ldots$$
$$= (l) \sum (-)^{l_1+l_2+\ldots-1} \frac{(l_1+l_2+\ldots-1)!}{l_1!\,l_2!\ldots} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

18. If we now multiply out the left-hand side in order to find the cofactor therein of

$$x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

we see that the cofactor consists of products of symmetric functions, and that each product is necessarily a separation of the symmetric function

$$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots).$$

Moreover, the coefficient of

$$x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots$$

in the product $X_{\mu_1}^{m_1} X_{\mu_2}^{m_2}\ldots$ is (vide first memoir loc. cit., p. 9)

$$\sum \frac{m_1!\,m_2!\ldots}{j_1!\,j_2!\ldots} (J_1)^{j_1}(J_2)^{j_2}\ldots,$$

wherein $(J_1)^{j_1}(J_2)^{j_2}\ldots$ is any separation of $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots)$ of specification $(\mu_1^{m_1}\mu_2^{m_2}\ldots)$.* Hence

$$(-)^{m_1+m_2+\ldots-1} \frac{(m_1+m_2+\ldots-1)!}{m_1!\,m_2!\ldots} X_{\mu_1}^{m_1} X_{\mu_2}^{m_2}\ldots$$

$$= \ldots + (-)^{m_1+m_2+\ldots-1} \frac{(m_1+m_2+\ldots-1)!}{m_1!\,m_2!\ldots} m_1!\,m_1!\ldots \left\{\sum \frac{(J_1)^{j_1}(J_2)^{j_2}\ldots}{j_1!\,j_2!\ldots}\right\} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots + \ldots$$

$$= \ldots + \left\{\sum (-)^{j_1+j_2+\ldots-1} \frac{(j_1+j_2+\ldots-1)!}{j_1!\,j_2!\ldots} (J_1)^{j_1}(J_2)^{j_2}\ldots\right\} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2}\ldots + \ldots,$$

* At Professor Cayley's suggestion, I abandon the expression "species partition" in favor of "specification," which is a far more appropriate word.

since $$m_1 + m_2 + \ldots . = j_1 + j_2 + \ldots .$$
Therefore

$$\sum\nolimits^{r} (-)^{l_1+l_2+\ldots .-1} \frac{(l_1+l_2+\ldots .-1)!}{l_1!\, l_2! \ldots .} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} \ldots .$$
$$= \sum \sum (-)^{j_1+j_2+\ldots .-1} \frac{(j_1+j_2+\ldots .-1)!}{j_1!\, j_2! \ldots .} (J_1)^{j_1} (J_2)^{j_2} \ldots . x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2} \ldots . ,$$

wherein $(J_1)^{j_1}(J_2)^{j_2} \ldots .$ is any separation of $(\lambda_1^{l_1}\lambda_2^{l_2} \ldots .)$.

Hence, substituting

$$\sum \sum (-)^{j_1+j_2+\ldots .-1} \frac{(j_1+j_2+\ldots .-1)!}{j_1!\, j_2! \ldots .} (J_1)^{j_1}(J_2)^{j_2} \ldots . x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2} \ldots .$$
$$= (l) \sum (-)^{l_1+l_2+\ldots .-1} \frac{(l_1+l_2+\ldots .-1)!}{l_1!\, l_2! \ldots .} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2} \ldots . ,$$

and equating coefficients of $x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2} \ldots .$ we obtain

$$(-)^{l_1+l_2+\ldots .} \frac{(l_1+l_2+\ldots .-1)!}{l_1!\, l_2! \ldots} (l)$$
$$= \sum (-)^{j_1+j_2+\ldots .-1} \frac{(j_1+j_2+\ldots .-1)!}{j_1!\, j_2! \ldots .} (J_1)^{j_1} (J_2)^{j_2} \ldots . ,$$

which is the theorem to be proved.

19. It will be observed that the theorem arises at once from the invariant property exhibited by the formula

$$\sum (-)^{l_1+l_2+\ldots .} \frac{(l_1+l_2+\ldots .-1)!}{l_1!\, l_2! \ldots .} X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} \ldots .$$
$$= (l) \sum (-)^{l_1+l_2+\ldots .} \frac{(l_1+l_2+\ldots .-1)!}{l_1!\, l_2! \ldots .} x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2} \ldots . ,$$

an application of the multinomial theorem in algebra being in reality all that is necessary; the formula in fact establishes the theorem at once for all partitions of all numbers, and is itself a condensed and exceedingly elegant analytical representation of it. It is very interesting to find an extensive proposition like the one under view appearing under the guise of an invariant of an algebraical transformation; I remark it particularly, as I have never met with a case at all similar to it before.

SECTION 3.

Property of the Coefficients of a Group.

20. On page 28 of my former memoir I defined a "Group" as applied to separations of a partition, and I recall here that the separation $(\lambda^2\mu)(\lambda)(\mu)$ belongs to the group
$$G\{(\lambda^2)(\lambda);\ (\mu)^2\}$$
because (λ^3) and (μ^2) occur in the separations $(\lambda^2)(\lambda)$ and $(\mu)^2$ respectively.

21. To put the group in evidence, it is expedient to substitute for the relations between $X_1, X_2, X_3, \ldots$ and $x_1, x_2, x_3, \ldots$ another set, as follows:

$$Y_1 = (1)\, y_1,$$
$$Y_2 = (2)\, y_2 + (1^2)\, y_{1^2},$$
$$Y_3 = (3)\, y_3 + (21)\, y_2 y_1 + (1^3)\, y_{1^3},$$
$$Y_4 = (4)\, y_4 + (31)\, y_3 y_1 + (2^2)\, y_{2^2} + (21^2)\, y_2 y_{1^2} + (1^4)\, y_{1^4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$Y_\mu = \Sigma\, (\mu_1^{m_1}\mu_2^{m_2}\ldots)\, y_{\mu_1{}^{m_1}} y_{\mu_2{}^{m_2}} \ldots ;$$

we then find

$$Y_4 - Y_3 Y_1 - \frac{1}{2} Y_2^2 + Y_2 Y_1^2 - \frac{1}{4} Y_1^4$$
$$= (4)\, y_4 + \{(31) - (3)(1)\} y_3 y_1 + (2^2)\, y_{2^2} - \frac{1}{2}(2)^2 y_2^2 + \{(21^2) - (2)(1^2)\}\, y_2 y_{1^2}$$
$$+ \{(2)(1)^2 - (21)(1)\}\, y_2 y_1^2 + (1^4)\, y_{1^4} - (1^3)(1)\, y_{1^3} y_1 - \frac{1}{2}(1^2)^2 y_{1^2}^2$$
$$+ (1^2)(1)^2 y_1^2 y_{1^2} - \frac{1}{4}(1)^4 y_1^4.$$

22. Observe that the cofactor of $y_3 y_1$ is composed of members of $G\{(3);(1)\}$,
" " " $y_2 y_{1^2}$ " " " $G\{(2);(1^2)\}$,
" " " $y_2 y_1^2$ " " " $G\{(2);(1)^2\}$,
and that these are the only y products which are multiplied by separations of a partition composed of different parts. Generally, in the cofactor of a y product,
$$y^{m_1'}_{\mu_1{}^{m_1}} y^{m_2'}_{\mu_2{}^{m_2}} \ldots ,$$
the equations must belong to the group
$$G\{(\mu_1^{m_1})^{m_1'};\ (\mu_2^{m_2})^{m_2'};\ldots\}.$$

23. I have before given the theorem that if the symmetric function (l) be expressed by means of separations of any partition of l which does not merely consist of repetitions of a single part, the algebraic sum of the coefficients of the separations of each group is zero. To establish this, it is merely necessary to

prove that, forming in succession

$$Y_1,$$
$$Y_2 - \frac{1}{2} Y_1^2,$$
$$Y_3 - Y_2 Y_1 + \frac{1}{3} Y_1^3,$$
$$Y_4 - Y_3 Y_1 - \frac{1}{2} Y_2^2 + Y_2 Y_1^2 - \frac{1}{4} Y_1^4,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

every product $$y^{m_1'}_{\mu_1 m_1} y^{m_2'}_{\mu_2 m_2}$$

vanishes on putting all the symmetric functions (1), (2), (1^2), (3), (21), (1^3), . . . equal to unity, unless $\mu_1 = \mu_2 = \ldots .$

24. For this purpose put

$$'Y_1 = y_1,$$
$$'Y_2 = y_2 + y_{1^2},$$
$$'Y_3 = y_3 + y_2 y_1 + y_{1^3},$$
$$'Y_4 = y_4 + y_3 y_1 + y_{2^2} + y_2 y_{1^2} + y_{1^4},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

so that

$$1 + 'Y_1 + 'Y_2 + 'Y_3 + \ldots .$$
$$= (1 + y_1 + y_{1^2} + y_{1^3} + \ldots .)(1 + y_2 + y_{2^2} + y_{2^3} + \ldots .)(1 + y_3 + y_{3^2} + y_{3^3} + \ldots .) \ldots$$

and taking logarithms

$$\log(1 + 'Y_1 + 'Y_2 + 'Y_3 + \ldots .) = \log(1 + y_1 + y_{1^2} + y_{1^3} + \ldots .)$$
$$+ \log(1 + y_2 + y_{2^2} + y_{2^3} + \ldots .) + \log(1 + y_3 + y_{3^2} + y_{3^3} + \ldots .) + \ldots .$$

and on expansion

$$'Y_1 + \left('Y_2 - \frac{1}{2}\,'Y_1^2\right) + \left('Y_3 - 'Y_2\,'Y_1 + \frac{1}{3}\,'Y_1^3\right)$$
$$+ \left('Y_4 - 'Y_3\,'Y_1 - \frac{1}{2}\,'Y_2^2 + 'Y_2\,'Y_1^2 - \frac{1}{4}\,'Y_1^4\right) + \ldots .$$
$$= y_1 + \left(y_{1^2} - \frac{1}{2} y_1^2\right) + \left(y_{1^3} - y_{1^2} y_1 + \frac{1}{3} y_1^3\right)$$
$$+ \left(y_{1^4} - y_{1^3} y_1 - \frac{1}{2} y_{1^2}^2 + y_{1^2} y_1^2 - \frac{1}{4} y_1^4\right) + \ldots .$$
$$+ y_2 + \left(y_{2^2} - \frac{1}{2} y_2^2\right) + \left(y_{2^3} - y_{2^2} y_2 + \frac{1}{3} y_2^3\right)$$
$$+ \left(y_{2^4} - y_{2^3} y_2 - \frac{1}{2} y_{2^2}^2 + y_{2^2} y_2^2 - \frac{1}{4} y_2^4\right) + \ldots .$$
$$+ y_3 + \left(y_{3^2} - \frac{1}{2} y_3^2\right) + \left(y_{3^3} - y_{3^2} y_3 + \frac{1}{3} y_3^3\right)$$
$$+ \left(y_{3^4} - y_{3^3} y_3 - \frac{1}{2} y_{3^2}^2 + y_{3^2} y_3^2 - \frac{1}{4} y_3^4\right) + \ldots .$$
$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

which may be written

$$\sum (-)^{l_1+l_2+\ldots\ldots-1} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2! \ldots\ldots} \, {}^{\prime}Y_{\lambda_1}^{l_1'} Y_{\lambda_2}^{l_2} \ldots\ldots$$

$$= \sum (-)^{l_1+l_2+\ldots\ldots-1} \frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\, l_2! \ldots\ldots} y_{1^{\lambda_1}}^{l_1} y_{1^{\lambda_2}}^{l_2} \ldots\ldots$$

$$+ \sum (-)^{l_1'+l_2'+\ldots\ldots-1} \frac{(l_1'+l_2'+\ldots\ldots-1)!}{l_1'!\, l_2'! \ldots\ldots} y_{2^{\lambda_1}}^{l_1'} y_{2^{\lambda_2}}^{l_2'} \ldots\ldots$$

$$+ \sum (-)^{l_1''+l_2''+\ldots\ldots-1} \frac{(l_1''+l_2''+\ldots\ldots-1)!}{l_1''!\, l_2''! \ldots\ldots} y_{3^{\lambda_1}}^{l_1''} y_{3^{\lambda_2}}^{l_2''} \ldots\ldots + \ldots\ldots$$

where the right-hand side, visibly, contains only products

$$y_{\mu_1^{m_1}}^{m_1'} y_{\mu_2^{m_2}}^{m_2'} \ldots\ldots,$$

in which $\mu_1 = \mu_2 = \ldots\ldots$

25. It is thus established that when we express the symmetric function (l) by means of separations of a partition of the number l, which does not merely consist of repetitions of a single part, the algebraic sum of the coefficients in each group of separations is zero.

This proof seems far preferable to the one given in the former memoir.

SECTION 4.

The Theory of Rational Symmetric Functions.

26. I propose to discuss symmetric functions which are rational, but are freed from the restriction of being integral.

Such an expression is $$\sum \frac{\alpha^p \beta^q}{\gamma^r} = \sum \alpha^p \beta^q \gamma^{-r};$$

attending merely to the indices, this may be written

$$(pq, -r),$$

in which form it appears as a partition with negative as well as positive parts. As far as I have discovered, Meyer Hirsch was the first who employed partitions with negative parts, but neither he nor any subsequent writer appears to have developed this part of the theory (vide Hirsch's Collection of Examples, Formu-

lae and Calculations on the Literal Calculus and Algebra, translated by Rev. J. A. Ross, London, 1827).

27. As a matter of convenience, I write the partition $(p, q, -r)$ in the form $(pq\bar{r})$, and writing the parts of such a partition in descending order of algebraical magnitude, thus:

$$(pq \ldots . \bar{r}\bar{s}).$$

28. I call p and s respectively the positive and negative degrees of the partition or of the symmetric function.

29. The sum $p+q+\ldots . -r-s$ is the weight of the partition or symmetric function, or quâ partitions it may be alluded to as the partible number.

30. Strictly speaking, the partition $(pq \ldots . \bar{r}\bar{s})$ may be spoken of as an algebraic partition of the partible number, but no confusion need arise in the comprehension of what follows if we speak merely of the partition instead of the algebraic partition.

31. For the sake of continuity, as well as for other weighty reasons which will appear, it is advisable to admit the zero as a possible part in such partitions. The general function to be studied then becomes

$$\sum \alpha^p \beta^q \ldots . \gamma^0 \delta^0 \ldots . \varepsilon^{-r} \theta^{-s},$$

which may be written

$$(pq \ldots . 00 \ldots . \bar{r}\bar{s}),$$

where $p, q, \ldots . r, s$ are integers.

32. Repetitions of the same part are as usual denoted by power indices, so that

$$(pp000\bar{r}\bar{r}\bar{r})$$

is written

$$(p^2 0^3 \bar{r}^3).$$

33. Regarding $p_1, p_2, p_3, \ldots . p_s$ as positive or negative integers excluding zero, we have evidently

$$(p_1 p_2 \ldots . p_s 0) = n-s . (p_1 p_2 \ldots . p_s),$$

$$(p_1 p_2 \ldots . p_s 0^2) = \frac{n-s . n-s-1}{1.2} (p_1 p_2 \ldots . p_s),$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

from which we obtain in succession

$$n(p_1p_2\ldots\ldots p_s) = s(p_1p_2\ldots\ldots p_s) + (p_1p_2\ldots\ldots p_s0),$$
$$n^2(p_1p_2\ldots\ldots p_s) = s^2(p_1p_2\ldots\ldots p_s) + \overline{2s+1}.(p_1p_2\ldots\ldots p_s0) + 2(p_1p_2\ldots\ldots p_s0^2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

so that the function $(p_1p_2\ldots\ldots p_s)$ multiplied by any rational integral algebraical function of n is expressible as a linear function of the expressions

$$(p_1p_2\ldots\ldots p_s),\quad (p_1p_2\ldots\ldots p_s0),\quad (p_1p_2\ldots\ldots p_s0^2),\ \ldots\ldots,$$

in which the coefficients are independent of n.

34. Hence we are considering symmetric functions which are rational algebraic functions of the n quantities

$$\alpha,\ \beta,\ \gamma,\ \ldots\ldots$$

and at the same time rational and integral algebraic functions of n.

35. Having in view a comprehensive study of the whole theory, I proceed as in the former case and put

$$\begin{aligned}
&1 + X_0\mu^0 + X_1\mu + X_2\mu^2 + \ldots\ldots \\
&\qquad + X_{-1}\frac{1}{\mu} + X_{-2}\frac{1}{\mu^2} + \ldots\ldots \\
&= \left(\begin{array}{l} 1 + \alpha^0x_0\mu^0 + \alpha x_1\mu + \alpha^2x_2\mu^2 + \ldots\ldots \\ \qquad + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2} + \ldots\ldots \end{array}\right) \\
&\times \left(\begin{array}{l} 1 + \beta^0x_0\mu^0 + \beta x_1\mu + \beta^2x_2\mu^2 + \ldots\ldots \\ \qquad + \frac{1}{\beta}x_{-1}\frac{1}{\mu} + \frac{1}{\beta^2}x_{-2}\frac{1}{\mu^2} + \ldots\ldots \end{array}\right) \\
&\times \left(\begin{array}{l} 1 + \gamma^0x_0\mu^0 + \gamma x_1\mu + \gamma^2x_2\mu^2 + \ldots\ldots \\ \qquad + \frac{1}{\gamma}x_{-1}\frac{1}{\mu} + \frac{1}{\gamma^2}x_{-2}\frac{1}{\mu^2} + \ldots\ldots \end{array}\right) \\
&\times \text{ etc.} \\
&= \prod_\alpha \left(\begin{array}{l} 1 + \alpha^0x_0\mu^0 + \alpha x_1\mu + \alpha^2x_2\mu^2 + \ldots\ldots \\ \qquad + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2} + \ldots\ldots \end{array}\right)
\end{aligned}$$

36. On multiplying out the right-hand side of this equation, the cofactor of μ^s (s positive, zero, or negative) is found to contain symmetric functions which are symbolical by all partitions of s into positive, zero and negative integers, and moreover, each of these symmetric functions (infinite in number) is attached to the corresponding x product.

Equating coefficients of like powers of μ, we obtain

$$\begin{aligned}
X_0 = {} & (0)\,x_0 + (1\bar{1})\,x_1x_{-1} + (2\bar{2})\,x_2x_{-2} + (2\bar{1}^2)\,x_2x_{-1}^2 \\
& \qquad + (1^2\bar{2})\,x_1^2x_{-2} + (1^2\bar{1}^2)\,x_1^2x_{-1}^2 + \ldots. \\
& + (0^2)\,x_0^2 + (10\bar{1})\,x_1x_0x_{-1} + (20\bar{2})\,x_2x_0x_{-2} + (20\bar{1}^2)\,x_2x_0x_{-1}^2 \\
& \qquad + (1^20\bar{2})\,x_1^2x_0x_{-2} + (1^20\bar{1}^2)\,x_1^2x_0x_{-1}^2 + \ldots. \\
& + (0^3)\,x_0^3 + (10^2\bar{1})\,x_1x_0^2x_{-1} + (20^2\bar{2})\,x_2x_0^2x_{-2} + (20^2\bar{1}^2)\,x_2x_0^2x_{-1}^2 \\
& \qquad + (1^20^2\bar{2})\,x_1^2x_0^2x_{-2} + (1^20^2\bar{1}^2)\,x_1^2x_0^2x_{-1}^2 + \ldots. \\
& + \ldots. + \ldots. \quad + \ldots. \quad + \ldots. + \ldots.
\end{aligned}$$

$$\begin{aligned}
X_1 = {} & (1)\,x_1 \qquad + (2\bar{1})\,x_2x_{-1} + (1^2\bar{1})\,x_1^2x_{-1} + \ldots. \\
& + (10)\,x_1x_0 + (20\bar{1})\,x_2x_0x_{-1} + (1^20\bar{1})\,x_1^2x_0x_{-1} + \ldots. \\
& + (10^2)\,x_1x_0^2 + (20^2\bar{1})\,x_2x_0^2x_{-1} + (1^20^2\bar{1})\,x_1^2x_0^2x_{-1} + \ldots. \\
& + \ldots. \quad + \ldots. \quad + \ldots.
\end{aligned}$$

$$\begin{aligned}
X_{-1} = {} & (\bar{1})\,x_{-1} \qquad + (1\bar{2})\,x_1x_{-2} + (1\bar{1}^2)\,x_1x_{-1}^2 + \ldots. \\
& + (0\bar{1})\,x_0x_{-1} + (10\bar{2})\,x_1x_0x_{-2} + (10\bar{1}^2)\,x_1x_0x_{-1}^2 + \ldots. \\
& + (0^2\bar{1})\,x_0^2x_{-1} + (10^2\bar{2})\,x_1x_0^2x_{-2} + (10^2\bar{1}^2)\,x_1x_0^2x_{-1}^2 + \ldots. \\
& + \ldots. \quad + \ldots. \quad + \ldots. \\
& \text{etc.}
\end{aligned}$$

and generally in the expression of X_s, s being positive, zero, or negative, the summation is taken for every partition of s into positive, zero, and negative integers.

37. Observe that we may write these relations in the form

$$\begin{aligned}
1 + X_0 = {} & (1+x_0)^n + (1\bar{1})\,x_1(1+x_0)^{n-2}x_{-1} + (2\bar{2})\,x_2(1+x_0)^{n-2}x_{-2} \\
& + (2\bar{1}^2)\,x_2(1+x_0)^{n-3}x_{-1}^2 + (1^2\bar{2})\,x_1^2(1+x_0)^{n-3}x_{-2} + (1^2\bar{1}^2)\,x_1^2(1+x_0)^{n-4}x_{-1}^2 \\
& + \ldots. \\
X_1 = {} & (1)\,x_1(1+x_0)^{n-1} + (2\bar{1})\,x_2(1+x_0)^{n-2}x_{-1} + (1^2\bar{1})\,x_1^2(1+x_0)^{n-3}x_{-1} + \ldots., \\
X_{-1} = {} & (\bar{1})(1+x_0)^{n-1}x_{-1} + (1\bar{2})\,x_1(1+x_0)^{n-2}x_{-2} + (1\bar{1}^2)\,x_1(1+x_0)^{n-3}x_{-1}^2 + \ldots., \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

38. And also in the forms

$$\frac{1+X_0}{(1+x_0)^n} = 1 + (1\bar{1})\frac{x_1x_{-1}}{(1+x_0)^2} + (2\bar{2})\frac{x_2x_{-2}}{(1+x_0)^2} + (2\bar{1}^2)\frac{x_2x_{-1}^2}{(1+x_0)^3}$$
$$+ (1^2\bar{2})\frac{x_1^2x_{-2}}{(1+x_0)^{n-3}} + (1^2\bar{1}^2)\frac{x_1^2x_{-1}^2}{(1+x_0)^4} + \ldots ,$$

$$\frac{X_1}{(1+x_0)^n} = (1)\frac{x_1}{1+x_0} + (2\bar{1})\frac{x_2x_{-1}}{(1+x_0)^2} + (1^2\bar{1})\frac{x_1^2x_{-1}}{(1+x_0)^3} + \ldots ,$$

$$\frac{X_{-1}}{(1+x_0)^n} = (\bar{1})\frac{x_{-1}}{1+x_0} + (1\bar{2})\frac{x_1x_{-2}}{(1+x_0)^2} + (1\bar{1}^2)\frac{x_1x_{-1}^2}{(1+x_0)^3} + \ldots$$

. . . .

39. These relations may be regarded as defining a transformation of

$$\begin{array}{lll} X_0,\ X_1,\ X_2,\ \ldots \text{ into functions of } & x_0,\ x_1,\ x_2,\ \ldots, \\ X_{-1},\ X_{-2},\ \ldots \quad \text{“} \quad \text{“} \quad \text{“} & x_{-1},\ x_{-2},\ \ldots, \end{array}$$

and we may seek the invariants of the transformation.

40. Recalling the relation

$$\begin{aligned} 1+X_0\mu^0+X_1\mu+X_2\mu^2+\ldots &= \prod_\alpha \{1+\alpha^0x_0\mu^0+\alpha x_1\mu+\alpha^2x_2\mu^2+\ldots\}, \\ +X_{-1}\frac{1}{\mu}+X_{-2}\frac{1}{\mu^2} \qquad & \qquad +\frac{1}{\alpha}x_{-1}\frac{1}{\mu}+\frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2}, \end{aligned}$$

and taking logarithms, we find

$$\begin{aligned} \log(1+X_0\mu^0+X_1\mu+X_2\mu^2+\ldots) &= \sum_\alpha \log(1+\alpha^0x_0\mu^0+\alpha x_1\mu+\alpha^2x_2\mu^2+\ldots), \\ +X_{-1}\frac{1}{\mu}+X_{-2}\frac{1}{\mu^2} \qquad & \qquad +\frac{1}{\alpha}x_{-1}\frac{1}{\mu}+\frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2}, \end{aligned}$$

which may be written

$$\log(1+X_0)+\log\left\{1+\frac{X_1}{1+X_0}\mu+\frac{X_2}{1+X_0}\mu^2+\ldots\right.$$
$$\left.+\frac{X_{-1}}{1+X_0}\frac{1}{\mu}+\frac{X_{-2}}{1+X_0}\frac{1}{\mu^2}+\ldots\right\}$$
$$=\sum_\alpha\left[\log(1+x_0)+\log\left\{1+\alpha\frac{x_1}{1+x_0}\mu+\alpha^2\frac{x_2}{1+x_0}\mu^2+\ldots\right.\right.$$
$$\left.\left.+\frac{1}{\alpha}\frac{x_{-1}}{1+x_0}\frac{1}{\mu}+\frac{1}{\alpha^2}\frac{x_{-2}}{1+x_0}\frac{1}{\mu^2}+\ldots\right\}\right]$$

41. We may now expand each side by the multinomial theorem and equate coefficients of like powers of μ.

Taking first the zero power of μ, we have

$$
\begin{aligned}
\log(1+X_0) &- \frac{X_1X_{-1}}{(1+X_0)^2} \\
&- \frac{X_2X_{-2}}{(1+X_0)^2} + \frac{X_1^2X_{-2}}{(1+X_0)^3} \\
&+ \frac{X_2X_{-1}^2}{(1+X_0)^3} - \frac{3}{2}\,\frac{X_1^2X_{-1}^2}{(1+X_0)^4} \\
&- \frac{X_3X_{-3}}{(1+X_0)^2} + 2\,\frac{X_2X_1X_{-3}}{(1+X_0)^3} - \frac{X_1^3X_{-3}}{(1+X_0)^4} \\
&+ 2\,\frac{X_3X_{-2}X_{-1}}{(1+X_0)^3} - 6\,\frac{X_2X_1X_{-2}X_{-1}}{(1+X_0)^4} + 4\,\frac{X_1^3X_{-2}X_{-1}}{(1+X_0)^5} \\
&- \frac{X_3X_{-1}^3}{(1+X_0)^4} + 4\,\frac{X_2X_1X_{-1}^3}{(1+X_0)^5} - \frac{10}{3}\,\frac{X_1^3X_{-1}^3}{(1+X_0)^6} \\
&+ \dots \\
= (0)\Big[\log(1+x_0) &- \frac{x_1x_{-1}}{(1+x_0)^2} \\
&- \frac{x_2x_{-2}}{(1+x_0)^2} + \frac{x_1^2x_{-2}}{(1+x_0)^3} \\
&+ \frac{x_2x_{-1}^2}{(1+x_0)^3} - \frac{3}{2}\,\frac{x_1^2x_{-1}^2}{(1+x_0)^4} \\
&- \frac{x_3x_{-3}}{(1+x_0)^2} + 2\,\frac{x_2x_1x_{-3}}{(1+x_0)^3} - \frac{x_1^3x_{-3}}{(1+x_0)^4} \\
&+ 2\,\frac{x_3x_{-2}x_{-1}}{(1+x_0)^3} + 6\,\frac{x_2x_1x_{-2}x_{-1}}{(1+x_0)^4} + 4\,\frac{x_1^3x_{-2}x_{-1}}{(1+x_0)^5} \\
&- \frac{x_3x_{-1}^3}{(1+x_0)^4} + 4\,\frac{x_2x_1x_{-1}^3}{(1+x_0)^5} - \frac{10}{3}\,\frac{x_1^3x_{-1}^3}{(1+x_0)^6} \\
&+ \dots\dots\dots\dots\Big],
\end{aligned}
$$

from which it appears that the left-hand side of the identity is an invariant of the transformation.

42. Observe that this invariant consists of a logarithmic term, together with an infinite succession of square blocks of terms; each of these blocks possesses row and column symmetry, both as regards the numerical coefficients and as regards the forms of the X products.

43. An invariant is likewise obtained from every other power of μ positive and negative, thus:

$$\begin{aligned}
&\frac{X_1}{1+X_0}\\
&-\frac{X_2X_{-1}}{(1+X_0)^2}+\frac{X_1^2X_{-1}}{(1+X_0)^3}\\
&-\frac{X_3X_{-2}}{(1+X_0)^2}+2\frac{X_2X_1X_{-2}}{(1+X_0)^3}-\frac{X_1^3X_{-2}}{(1+X_0)^4}\\
&+\frac{X_3X_{-1}^2}{(1+X_0)^3}-3\frac{X_2X_1X_{-1}^2}{(1+X_0)^4}+2\frac{X_1^3X_{-1}^2}{(1+X_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
=(1)\Big[&\frac{x_1}{1+x_0}\\
&-\frac{x_2x_{-1}}{(1+x_0)^2}+\frac{x_1^2x_{-1}}{(1+x_0)^3}\\
&-\frac{x_3x_{-2}}{(1+x_0)^2}+2\frac{x_2x_1x_{-2}}{(1+x_0)^3}-\frac{x_1^3x_{-2}}{(1+x_0)^4}\\
&+\frac{x_3x_{-1}^2}{(1+x_0)^3}-3\frac{x_2x_1x_{-1}^2}{(1+x_0)^4}+2\frac{x_1^3x_{-1}^2}{(1+x_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big],
\end{aligned}$$

and

$$\begin{aligned}
&\frac{X_{-1}}{1+X_0}\\
&-\frac{X_1X_{-2}}{(1+X_0)^2}+\frac{X_1X_{-1}^2}{(1+X_0)^3}\\
&-\frac{X_2X_{-3}}{(1+X_0)^2}+2\frac{X_2X_{-1}X_{-2}}{(1+X_0)^3}-\frac{X_2X_{-1}^3}{(1+X_0)^4}\\
&+\frac{X_1^2X_{-3}}{(1+X_0)^3}-3\frac{X_1^2X_{-1}X_{-2}}{(1+X_0)^4}+2\frac{X_1^2X_{-1}^3}{(1+X_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
=(\bar{1})\Big[&\frac{x_{-1}}{1+x_0}\\
&-\frac{x_1x_{-2}}{(1+x_0)^2}+\frac{x_1x_{-1}^2}{(1+x_0)^3}\\
&-\frac{x_2x_{-3}}{(1+x_0)^2}+2\frac{x_2x_{-1}x_{-2}}{(1+x_0)^3}-\frac{x_2x_{-1}^3}{(1+x_0)^4}\\
&+\frac{x_1^2x_{-3}}{(1+x_0)^3}-3\frac{x_1^2x_{-1}x_{-2}}{(1+x_0)^4}+2\frac{x_1^2x_{-1}^3}{(1+x_0)^5}\\
&+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big],
\end{aligned}$$

and so forth.

44. These invariants may be written

$$\log(1+X_0)+\sum(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}=\left(\frac{X_{\lambda_1}}{1+X_0}\right)^{l_1}\left(\frac{X_{\lambda_2}}{1+X_0}\right)^{l_2}\ldots,$$

the summation being for all solutions of the equation

$$l_1\lambda_1+l_2\lambda_2+\ldots\ldots=0,$$

in positive and negative integers, but excluding zero; and

$$\sum(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}\left(\frac{X_{\lambda_1}}{1+X_0}\right)^{l_1}\left(\frac{X_{\lambda_2}}{1+X_0}\right)^{l_2}\ldots\ldots,$$

where the summation is for all integer and non-zero solutions of the equation

$$l_1\lambda_1+l_2\lambda_2+\ldots\ldots=m,$$

m being any positive or negative but not a zero integer.

45. We may expand the logarithm in the invariant of weight zero, and moreover in all the invariants we may expand the factors

$$\left(\frac{1}{1+X_0}\right)^{l_1},\ \left(\frac{1}{1+X_0}\right)^{l_2},\ldots\ldots,$$

and we see that we may write the whole system of invariants in the form

$$\sum(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}\ldots\ldots,$$

where now $\qquad l_1\lambda_1+l_2\lambda_2+\ldots\ldots=m;$

m may be any integer, positive, zero or negative, and the summation is in regard to all solutions of the indeterminate equation

$$l_1\lambda_1+l_2\lambda_2+\ldots\ldots=m,$$

in positive, zero and negative integers.

46. We can now enunciate as follows:

Theorem. If

$$\begin{aligned}1+X_0+X_1\ &+X_2+\ldots\ldots=\prod_{\alpha}(1+\alpha^0x_0+\alpha x_1\quad+\alpha^2x_2+\ldots\ldots),\\ +X_{-1}&+X_{-2}\qquad\qquad\qquad\qquad\qquad\quad+\frac{1}{\alpha}x_{-1}+\frac{1}{\alpha^2}x_{-2}\end{aligned}$$

then

$$\sum(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}\ldots\ldots$$

$$=(m)\sum(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}x_{\lambda_1}^{l_1}x_{\lambda_2}^{l_2}\ldots\ldots,$$

where the summations are for all solutions of the indeterminate equation

$$l_1\lambda_1 + l_2\lambda_2 + \ldots\ldots = m,$$

in positive, zero, and negative integers, and m is any integer, positive, zero, or negative.

47. This invariant property that has just been established is fundamental and of very great importance.

We now proceed as in the previous more simple case, to multiply out the sinister of the identity, in order to find out therein the cofactor of $x_{\lambda_1}^{l_1} x_{\lambda_2}^{l_2} \ldots\ldots$; this cofactor is an assemblage of symmetric function products, each of which is symbolized by a separation of the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$, and we obtain the numerical coefficients by application of the ordinary multinomial theorem: the reasoning is the same as in the previous case, and we are thus led to the comprehensive theorem

$$(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}(\dot{m})$$
$$= \sum (-)^{j_1+j_2+\ldots\ldots-1}\frac{(j_1+j_2+\ldots\ldots-1)!}{j_1!\,j_2!\ldots\ldots}(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots,$$

or, as this may be written,

48. $$(-)^{l_1+l_2+\ldots\ldots-1}\frac{(l_1+l_2+\ldots\ldots-1)!}{l_1!\,l_2!\ldots\ldots}S(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$$
$$= \sum (-)^{j_1+j_2+\ldots\ldots-1}\frac{(j_1+j_2+\ldots\ldots-1)!}{j_1!\,j_2!\ldots\ldots}(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots,$$

wherein $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$ is any partition of $m\,(=\Sigma l\lambda)$ into positive, zero and negative integers; $S(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$ denotes the symmetric function (m) expressed by means of separations of the partition $(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots)$ of the number m; $(J_1)^{j_1}(J_2)^{j_2}\ldots\ldots$ is any separation of the partition

$$(\lambda_1^{l_1}\lambda_2^{l_2}\ldots\ldots);$$

and the summation is in regard to all such separations. Two examples of this theorem are subjoined.

Example I.

49. To express the symmetric function (2) by means of separations of the symmetric function

$$(210\bar{1}).$$

We form two columns, the first consisting of the different separations, and the second involving the coefficients given by the theorem. We thus have

Separable Partition	Coefficient
$(210\bar{1})$	-6

Separations	Coefficients
$(2)(1)(0)(\bar{1})$	-6
$(21)(0)(\bar{1})$	$+2$
$(20)(1)(\bar{1})$	$+2$
$(2\bar{1})(1)(0)$	$+2$
$(10)(2)(\bar{1})$	$+2$
$(1\bar{1})(2)(0)$	$+2$
$(0\bar{1})(2)(1)$	$+2$
$(21)(0\bar{1})$	-1
$(20)(1\bar{1})$	-1
$(2\bar{1})(10)$	-1
$(210)(\bar{1})$	-1
$(21\bar{1})(0)$	-1
$(20\bar{1})(1)$	-1
$(10\bar{1})(2)$	-1
$(210\bar{1})$	$+1$

Hence

$$\begin{aligned}
-6(2) = &-6(2)(1)(0)(\bar{1}) \\
&+2\{(21)(0)(\bar{1})+(20)(1)(\bar{1})+(2\bar{1})(1)(0)+(10)(2)(\bar{1}) \\
&\qquad\qquad +(1\bar{1})(2)(0)+(0\bar{1})(2)(1)\} \\
&-\{(21)(0\bar{1})+(20)(1\bar{1})+(2\bar{1})(10\} \\
&-\{(210)(\bar{1})+(21\bar{1})(0)+(20\bar{1})(1)+(10\bar{1})(2)\} \\
&+(210\bar{1}).
\end{aligned}$$

50. To verify this identity, observe that

$$\begin{aligned}
(0) &= n, \\
(\lambda 0) &= n-1.(\lambda), \\
(\lambda\mu 0) &= n-2.(\lambda\mu),
\end{aligned}$$

so that the identity leads to

$$
\begin{aligned}
-6(2) = & -6n(2)(1)(\bar{1}) \\
& + 2n\{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
& + 6(n-1)(2)(1)(\bar{1}) \\
& - (n-1)\{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
& - n(21\bar{1}) \\
& - (n-2)\{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
& + (n-3)(21\bar{1});
\end{aligned}
$$

which reduces to

$$
\begin{aligned}
+2(2) = & +2(2)(1)(\bar{1}) \\
& - \{(21)(\bar{1}) + (2\bar{1})(1) + (1\bar{1})(2)\} \\
& + (21\bar{1}),
\end{aligned}
$$

a result which is precisely that given by the theorem for the expression of (2) by means of separations of

$$(21\bar{1}).$$

51. Written in the algebraic form, this last result is

$$2\Sigma\alpha^2 = 2\Sigma\alpha^2\Sigma\alpha'\Sigma\alpha^{-1} - \Sigma\alpha^2\beta'\Sigma\alpha^{-1} - \Sigma\alpha^2\beta^{-1}\Sigma\alpha' - \Sigma\alpha'\beta^{-1}\Sigma\alpha^2 + \Sigma\alpha^2\beta'\gamma^{-1}.$$

52. Example II.

To express S_3 by means of separations of $(3^3\bar{3}^2)$. The result arranged by groups is as follows:

$$
\begin{array}{llllllll}
2S_3 = & 2(3)^3(\bar{3})^2 & -(3)^3(\bar{3}^2) & -3(3^2)(3)(\bar{3})^2 & +2(3^2)(3)(\bar{3}^2) & +(3^3)(\bar{3})^2 & -(3^3)(\bar{3}^2) \\
 & & & & & & -(3^3)(\bar{3}^2) \\
 & -3(3)^2(3\bar{3})(\bar{3}) & +(3)^2(3\bar{3}^2) & +2(3^2\bar{3})(3)(\bar{3}) & -(3^2\bar{3}^2)(3) & -(3^3\bar{3})(\bar{3}) & +(3^3\bar{3}^2) \\
 & +(3)(3\bar{3})^2 & & +2(3^2)(3\bar{3})(\bar{3}) & -(3^2)(3\bar{3}^2) & & \\
 & & & -(3^2\bar{3})(3\bar{3}) & & & \\
\hline
 & \pm 3 & \pm 1 & \pm 4 & \pm 2 & \pm 1 & \pm 1 \\
\hline
\end{array}
$$

53. To establish the law that the algebraic sum of the coefficients in each group is constant, we proceed, as in paragraph 24, and put

$$
\begin{aligned}
1 + {}'Y_0 + {}'Y_1 + {}'Y_2 + \ldots\ldots & \\
= (1 + y_0 + y_{0^2} + y_{0^3} + \ldots)(1 + y_1 + y_{1^2} + y_{1^3} + \ldots)(1 + y_2 + y_{2^2} + y_{2^3} + \ldots)\ldots & \\
+ {}'Y_{-1} + {}'Y_{-2} \qquad \times (1 + y_{\bar{1}} + y_{\bar{1}^2} + y_{\bar{1}^3} + \ldots)(1 + y_{\bar{2}} + y_{\bar{2}^2} + y_{\bar{2}^3} + \ldots)\ldots &
\end{aligned}
$$

Now, taking logarithms, the demonstration proceeds *pari passu* with the former and simpler case.

Section 5.

The Law of Reciprocity.

54. I pass on to the generalization of the law of reciprocity which was established in the former memoir, p. 3 et seq.

55. The theorem to be proved is:

Theorem. "Writing

$$\begin{aligned}1 + X_0\mu^0 + X_1\mu + X_2\mu^2 + \ldots & = \prod_\alpha \left(1 + \alpha^0 x_0\mu^0 + \alpha x_1\mu + \alpha^2 x_2\mu^2 + \ldots\right),\\ + X_{-1}\frac{1}{\mu} + X_{-2}\frac{1}{\mu^2} \quad & \qquad + \frac{1}{\alpha}x_{-1}\frac{1}{\mu} + \frac{1}{\alpha^2}x_{-2}\frac{1}{\mu^2},\end{aligned}$$

where the product extends to each of the n quantities

$$\alpha, \beta, \gamma, \ldots \qquad (n = \infty),$$

and forming and developing the product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots,$$

we obtain a result

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots = \ldots + \theta\left(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\right)x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots,$$

θ being the numerical coefficient of the term

$$\left(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\ldots\right)x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots,$$

in the development of the product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\ldots;$$

then $\quad X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\ldots = \ldots + \theta\left(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\right)x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots + \ldots,$

that is to say, the coefficient of the term

$$\left(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\right)x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots$$

in the development of the product

$$X_{\lambda_1}^{l_1}X_{\lambda_2}^{l_2}X_{\lambda_3}^{l_3}\ldots$$

is the same number θ."

56. The proof here presented is, as was the one in the former memoir, purely arithmetical in its nature, and depends upon the consideration of a particular mode of distribution of a given number of objects into the same number

of parcels, no parcel being empty; we have invariably one object in each parcel. The distribution is of a more general character than the one previously considered and includes the latter as a particular case. It will be seen that when once the character of the distribution has been precisely defined and its connection, with the subject treated of, established by a close examination of a particular case, the actual proof is instantaneous; it arises in fact from a single observation which is of such an elementary character that it admits of no dispute.

It is necessary to make some definitions more extended than those given in Proc. Lond. Math. Soc., Vol. XIX, p. 243.

57. Suppose any number of objects, all of the same kind, to be separated into an upper group and a lower group, in such wise that the upper group consists of λ_1 more objects than the lower group; such an assemblage of similar objects, so separated, may be spoken of as "Objects of type (λ_1)"; the actual number of objects is immaterial; so long as the number of objects in the upper group exceeds the number in the lower group by λ_1, the objects are of type (λ_1).

58. The number λ_1 may be positive, zero, or negative.

Ex. gr. Objects of type (0) may be $\begin{matrix} a \\ a \end{matrix}$ or $\begin{matrix} aaa \\ aaa \end{matrix}$ or etc.

and objects of type $(\bar{2})$ may be $\begin{matrix} \\ aa \end{matrix}$ or $\begin{matrix} a \\ aaa \end{matrix}$ or $\begin{matrix} aaa \\ aaaaa \end{matrix}$ or etc.

59. I make a distinction between "Objects of type (λ_1)" and "Objects (λ_1)."

I consider "Objects of *type* (λ_1)" to have reference to objects of any, the same, kind, so that objects

$$\begin{matrix} a \\ a \end{matrix} \text{ or } \begin{matrix} b \\ b \end{matrix}, \text{ etc.}$$

are alike of type (0); whereas, when the objects are restricted to be of a certain definite kind a, I speak of "Objects (λ_1)."

60. Again: "Objects of type $(\lambda_1\lambda_2\lambda_3 \ldots .)$" is defined to mean

(i). "Objects of type (λ_1) of one kind.
(ii). "Objects of type (λ_2) of a second kind.
(iii). "Objects of type (λ_3) of a third kind.
. . . . ;

thus "objects of type $(30\bar{1})$" may be such as

$$\begin{matrix} ccc \\ \end{matrix} \quad \begin{matrix} ddd \\ ddd \end{matrix} \quad \begin{matrix} ee \\ eee \end{matrix} \text{ or } \begin{matrix} eeee \\ e \end{matrix} \quad \begin{matrix} f \\ f \end{matrix} \quad \begin{matrix} g \\ gg \end{matrix},$$

where the species of object obtaining in each group is not specified; whereas, if it be stated or implied that the objects in the three groups are of given species, say a, b, c respectively, we would speak of "objects $(30\bar{1})$"; then "objects $(30\bar{1})$" might mean an assemblage such as

$$\begin{array}{lll} aaaaa & bb & \\ aa & bb & c \end{array},$$

the excesses of the objects in the upper group over those in the lower groups being respectively 3, 0 and -1.

The distinction made between "objects of type $(\lambda_1\lambda_2\lambda_3 \ldots .)$" and "objects $(\lambda_1\lambda_2\lambda_3 \ldots .)$" will be now understood.

61. Observe that "objects (0)" refers to a set of at least two objects, one in each group.

62. If no restriction be placed upon the number of objects, there is an infinite number of assemblages included in the phrase "objects $(\lambda_1\lambda_2\lambda_3 \ldots .)$"; by fixing the number of objects we obtain a finite number of assemblages; fixing the number of objects at 8, "objects $(30\bar{1})$" will comprise the three assemblages:

$$\begin{array}{lll} aaa & b & c \\ & b & cc \end{array}; \quad \begin{array}{lll} aaa & bb & \\ & bb & c \end{array}; \quad \begin{array}{lll} aaaa & b & \\ a & b & c \end{array}$$

63. We have now objects of various kinds, divided into upper and lower groups, and we may have boxes or parcels of various kinds, similarly divided into certain upper and lower groups, to contain these objects in such wise that one parcel of an upper group contains one object of an upper group, and one parcel of a lower group contains one object of a lower group, there being as many parcels as objects.

64. "Parcels of type $(\lambda_1\lambda_2\lambda_3 \ldots .)$" and "Parcels $(\lambda_1\lambda_2\lambda_3 \ldots .)$" are defined precisely as in the case of objects, capital letters being employed to exhibit them instead of small ones.

65. Thus 9 "Parcels $(10^2\bar{2})$" will comprise the four assemblages of parcels:

$$\begin{array}{llll} A & B & C & D \\ & B & C & DDD \end{array}; \quad \begin{array}{llll} A & BB & C & \\ & BB & C & DD \end{array};$$

$$\begin{array}{llll} A & B & CC & \\ & B & CC & DD \end{array}; \quad \begin{array}{llll} AA & B & C & \\ A & B & C & DD \end{array}$$

66. Let us now take 8 "objects $(30\bar{1})$," viz. the three assemblages

$$\begin{array}{lll} aaa & b & c \\ & b & cc \end{array}; \quad \begin{array}{lll} aaa & bb & \\ & bb & c \end{array}; \quad \begin{array}{lll} aaaa & b & \\ a & b & c \end{array};$$

and also 8 "parcels $(4\bar{2})$," viz. the two assemblages

$$\begin{array}{ll} AAAAA & \\ A & BB \end{array}; \quad \begin{array}{ll} AAAA & B \\ & BBB \end{array}.$$

We make a distribution of 8 "objects $(30\bar{1})$" into 8 "parcels $(4\bar{2})$" by placing the objects which occur in any one of the assemblages of objects into the parcels which occur in any one of the assemblages of parcels, in such wise that objects of upper and lower groups appear only in parcels of upper and lower groups respectively, and one parcel contains one and only one object.

67. This distribution is practicable because the partitions

$$(30\bar{1}) \text{ and } (4\bar{2})$$

are each of the same weight, viz. 2. In this manner a definite number of distributions is obtained.

Let us place the second assemblage of objects in the first assemblage of parcels: thus, as one case, we have

$$\begin{array}{ll} AAAAA & \\ aaabb & \\ A & BB \\ b & bc \end{array}$$

68. An examination of this distribution shows us that we can separate it into four portions, so that each portion consists of but one kind of parcel and of but one kind of object; the four portions are

$$\left|\begin{array}{c|c|c|c} \mathrm{I} & \mathrm{II} & \mathrm{III} & \mathrm{IV} \\ AAA & AA & & \\ aaa & bb & & \\ & A & B & B \\ & b & b & c \end{array}\right|,$$

wherein portion I contains "objects of type (3)" placed in "parcels of type (3),"
II " " " (1)" " " " " (1),"
III " " " $(\bar{1})$" " " " " $(\bar{1})$,"
IV " " " $(\bar{1})$" " " " " $(\bar{1})$";

this particular case of distribution possesses a property which is indicated by the succession of numbers 3, 1, -1, -1; thus the property may be defined by the partition $(31\bar{1}^2)$ whose weight is 2, which is of necessity the same as that common to the partitions $(30\bar{1})$, $(4\bar{2})$ which define the assemblages of objects and parcels.

69. We may now restrict ourselves to those distributions of assemblages of objects into assemblages of parcels which possess the property defined by the partition $(31\bar{1}^2)$.

70. This partition $(31\bar{1}^2)$ will be spoken of as the "partition of restriction."

71. The whole number of distributions of assemblages of objects $(30\bar{1})$ into assemblages of parcels $(4\bar{2})$, subject to the restriction of partition $(31\bar{1}^2)$, are now given; they are four in number, viz.

$$\left\{\begin{array}{ll} AAAAA & \\ aaabb & \\ A & BB \\ b & bc \end{array}\right. ,$$

$$\left\{\begin{array}{ll} AAAAA & \\ aaaab & \\ A & BB \\ a & bc \end{array}\right. ,$$

$$\left\{\begin{array}{ll} AAAA & B \\ aaab & c \\ & BBB \\ & bcc \end{array}\right. ,$$

$$\left\{\begin{array}{ll} AAAA & B \\ aaab & b \\ & BBB \\ & bbc \end{array}\right.$$

72. It is to be understood that the distributions now under examination are connected with three partitions of the same number; the partition of the objects, the partition of the parcels, and the partition of restriction.

73. The weight of the partitions may be any integer, positive, zero, or negative.

74. The number of objects may be any whatever, subject merely to a lower limit which is fixed by the partition of restriction; if a positive part λ occur in this partition, λ objects at least are thereby implied; a negative part $\bar{\lambda}$ also implies at least λ objects, whilst each part zero necessitates at least two objects; thus if p be the sum of the positive parts, if there be q zeros, and if r be the sum of the negative parts,

$$p + 2q + r$$

is a lower limit to the number of objects which can be taken, while in general we may take $p + 2q + r + 2m$ objects, where m is zero or any positive integer.

75. For present purposes it is necessary to consider a minimum number of objects as taking part in the distributions; this, as above mentioned, is known as soon as we decide upon the partition of restriction.

In the example already given, 8 objects were taken, but 6 objects may be taken, as is evident from the partition of restriction $(31\bar{1}^2)$. Reducing the number to 6, we find one assemblage of objects

$$\begin{array}{lll} aaa & b & \\ & b & c \end{array}$$

and also one assemblage of parcels

$$\begin{array}{ll} AAAA & \\ & BB \end{array}$$

and subject to the restriction, but one distribution, viz.

$$\left\{\begin{array}{ll} AAAA & \\ a\,a\,a\,b & \\ & BB \\ & b\,c \end{array}\right.$$

76. In general, therefore, our distributions are precisely defined by three partitions of the same number, and in every case their number will be perfectly definite.

77. It is now necessary to make a minute examination of a particular case of the general theorem, in order that we may see the bearing of this theory of distribution upon the multiplication of symmetric functions.

78. Since

$$\begin{aligned}X_1 = {} & (1)\,x_1 + (10)\,x_1x_0 + (10^2)\,x_1x_0^2 + \dots \\ & + (2\bar{1})\,x_2x_{-1} + (20\bar{1})\,x_2x_0x_{-1} + (20^2\bar{1})\,x_2x_0^2x_{-1} + \dots \\ & + (1^2\bar{1})\,x_1^2x_{-1} + (1^20\bar{1})\,x_1^2x_0x_{-1} + (1^20^2\bar{1})\,x_1^2x_0^2x_{-1} + \dots \\ & + (3\bar{2})\,x_3x_{-2} + (30\bar{2})\,x_3x_0x_{-2} + (30^2\bar{2})\,x_3x_0^2x_{-2} + \dots \\ & + (3\bar{1}^2)\,x_3x_{-1}^2 + (30\bar{1}^2)\,x_3x_0x_{-1}^2 + (30^2\bar{1}^2)\,x_3x_0^2x_{-1}^2 + \dots \\ & + (21\bar{2})\,x_2x_1x_{-2} + (210\bar{2})\,x_2x_1x_0x_{-2} + (210^2\bar{2})\,x_2x_1x_0^2x_{-2} + \dots \\ & + (21\bar{1}^2)\,x_2x_1x_{-1}^2 + (210\bar{1}^2)\,x_2x_1x_0x_{-1}^2 + (210^2\bar{1}^2)\,x_2x_1x_0^2x_{-1}^2 + \dots \\ & + (1^3\bar{2})\,x_1^3x_{-2} + (1^30\bar{2})\,x_1^3x_0x_{-2} + (1^30^2\bar{2})\,x_1^3x_0^2x_{-2} + \dots \\ & + (1^3\bar{1}^2)\,x_1^3x_{-1}^2 + (1^30\bar{1}^2)\,x_1^3x_0x_{-1}^2 + (1^30^2\bar{1}^2)\,x_1^3x_0^2x_{-1}^2 + \dots \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

$$\begin{aligned}X_{-2} = {} & (\bar{2})\,x_{-2} + (0\bar{2})\,x_0x_{-2} + (0^2\bar{2})\,x_0^2x_{-2} + \dots \\ & + (1\bar{3})\,x_1x_{-3} + (10\bar{3})\,x_1x_0x_{-3} + (10^2\bar{3})\,x_1x_0^2x_{-3} + \dots \\ & + (\bar{1}^2)\,x_{-1}^2 + (0\bar{1}^2)\,x_0x_{-1}^2 + (0^2\bar{1}^2)\,x_0^2x_{-1}^2 + \dots \\ & + (1\bar{1}\bar{2})\,x_1x_{-1}x_{-2} + (10\bar{1}\bar{2})\,x_1x_0x_{-1}x_{-2} + (10^2\bar{1}\bar{2})\,x_1x_0^2x_{-1}x_{-2} + \dots \\ & + (1\bar{1}^3)\,x_1x_{-1}^3 + (10\bar{1}^3)\,x_1x_0x_{-1}^3 + (10^2\bar{1}^3)\,x_1x_0^2x_{-1}^3 + \dots \\ & + \dots\dots\dots\dots\dots\dots\dots\dots\end{aligned}$$

we have

$$X_1X_{-2} = \dots + \{(1^2\bar{1})(0\bar{2}) + (1^20\bar{1})(\bar{2}) + (1)(10\bar{1}\bar{2}) + (10)(1\bar{1}\bar{2})\}\,x_1^2x_0x_{-1}x_{-2} + \dots$$

79. The partition of the term $x_1^2x_0x_{-1}x_{-2}$ is $(1^20\bar{1}\bar{2})$; each of the products

$$(1^2\bar{1})(0\bar{2}),\quad (1^20\bar{1})(\bar{2}),\quad (1)(10\bar{1}\bar{2}),\quad (10)(1\bar{1}\bar{2}),$$

is a separation of the partition $(1^20\bar{1}\bar{2})$ of specification $(1\bar{2})$; this follows of necessity because $(1\bar{2})$ is the partition of the term X_1X_{-2}.

80. When the products which occur in the coefficient of the term $x_1^2x_0x_{-1}x_{-2}$ are multiplied, a monomial symmetric function $(1^2\bar{1}\bar{2})$ will be presented attached to a certain numerical coefficient; supposing the symmetric functions refer to quantities a, b, c, ..., we have

$$(1^2\bar{1})(0\bar{2}) = \sum \frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\sum\frac{a}{a}\cdot\frac{1}{b^2},$$

and also

$$(1^2\bar{1}\bar{2}) = \sum \frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}.$$

81. A term $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ of the symmetric function $(1^2\overline{1}\overline{2})$ arises from the multiplication $(1^2\overline{1})(0\overline{2})$ in the three ways:

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{a}{a}\cdot\frac{1}{d^2}\right),$$

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{b}{b}\cdot\frac{1}{d^2}\right),$$

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{c}{c}\cdot\frac{1}{d^2}\right),$$

for each of the terms $\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$, $\frac{a}{1}\cdot\frac{b^2}{b}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$, $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{c}{c^2}\cdot\frac{1}{d^2}$, is the same term $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ of the function $(1^2\overline{1}\overline{2})$.

82. Observe that such a product as

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{e}{e}\cdot\frac{1}{d^2}\right)$$

gives rise to a term $\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{e}{e}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ which belongs to the function $(1^2 0\overline{1}\overline{2})$ and not to $(1^2\overline{1}\overline{2})$; the coefficient of $(1^2\overline{1}\overline{2})$ in the product $(1^2\overline{1})(0\overline{2})$ is thus 3.

83. To connect this result with the preceding theory of distribution, observe that the terms

$$\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2},\quad \frac{a}{1}\cdot\frac{b^2}{b}\cdot\frac{1}{c}\cdot\frac{1}{d^2},\quad \frac{a}{1}\cdot\frac{b}{1}\cdot\frac{c}{c^2}\cdot\frac{1}{d^2},$$

may each be considered as representing an assemblage of 7 objects $(1^2\overline{1}\overline{2})$, the numerator and denominator letters denoting objects in the upper and lower groups respectively; the term $\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ arose from the multiplication

$$\left(\frac{a}{1}\cdot\frac{b}{1}\cdot\frac{1}{c}\right)\left(\frac{a}{a}\cdot\frac{1}{d^2}\right),$$

and, conversely, we may regard it as decomposed in this manner; we may further consider this decomposition of the term $\frac{a^2}{a}\cdot\frac{b}{1}\cdot\frac{1}{c}\cdot\frac{1}{d^2}$ to denote a distribution of the assemblage of 7 objects represented by the term; this distribu-

tion will be into an assemblage of 7 parcels $(1\bar{2})$ and will be indicated by the scheme

$$\begin{array}{ll} AA & B \\ ab & a \\ A & BBB \\ c & add \end{array}$$

84. Drawing a vertical line between the A and the B parcels, the scheme breaks up into two portions; the left-hand portion denotes a distribution of objects $(1^2\bar{1})$ into parcels (1), whilst the right-hand shows a distribution of objects $(0\bar{2})$ into parcels $(\bar{2})$; the distribution is of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, and it is necessarily subject to a restriction whose partition is $(1^20\bar{1}\bar{2})$ because the term $x_1^2x_0x_{-1}x_{-2}$ has this partition; the two portions into which the distribution may be divided are respectively restricted by partitions $(1^2\bar{1})$ and $(0\bar{2})$ because these partitions are factors of the separation

$$(1^2\bar{1})(0\bar{2})$$

which is being discussed.

85. Two more distributions of precisely the same nature correspond to the two terms

$$\frac{a}{1}\cdot\frac{b^2}{b}\cdot\frac{1}{c}\cdot\frac{1}{d^2}, \quad \frac{a}{1}\cdot\frac{b}{1}\cdot\frac{c}{c^2}\cdot\frac{1}{d^2};$$

these are

$$\begin{array}{ll} AA & B \\ ab & b \\ A & BBB \\ c & bdd \end{array}, \quad \begin{array}{ll} AA & B \\ ab & c \\ A & BBB \\ c & cdd \end{array};$$

each of the three distributions is of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, and is not only subject to the restriction whose partition is $(1^20\bar{1}\bar{2})$, but also more minutely to the compound restriction indicated by the separation $(1^2\bar{1})(0\bar{2})$.

86. It is thus clear that, corresponding to the algebraical result

$$(1^2\bar{1})(0\bar{2}) = \ldots\ldots + 3(1^2\bar{1}\bar{2}) + \ldots\ldots,$$

we have a distribution theorem, viz.

"There are 3 ways of distributing objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, $(1\bar{2})$ being the specification of the separation $(1^2\bar{1})(0\bar{2})$, subject to the compound restriction, of separation $(1^2\bar{1})(0\bar{2})$."

87. Consider next the product

$$(1^2 0 \bar{1})(\bar{2}) = \sum \frac{a}{1} \cdot \frac{b}{1} \cdot \frac{c}{c} \cdot \frac{1}{d} \sum \frac{1}{a^2};$$

the term $\frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}$ can only arise from the product

$$\left(\frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{d}{d}\right)\left(\frac{1}{d^2}\right);$$

the coefficient of $(1^2 \bar{1} \bar{2})$ in the product $(1^2 0 \bar{1})(\bar{2})$ is therefore unity; the corresponding distribution is seen to be

$$\begin{array}{ll} AAA & \\ abd & \\ AA & BB \\ cd & dd \end{array};$$

the restrictions in the A and B parcels are respectively $(1^2 0 \bar{1})$ and $(\bar{2})$; hence we have a distribution of objects $(1^2 \bar{1} \bar{2})$ into parcels $(1 \bar{2})$ subject to the composite restriction $(1^2 0 \bar{1})(\bar{2})$.

88. Again the product

$$(1)(10\bar{1}\bar{2}) = \sum \frac{a}{1} \sum \frac{a}{1} \cdot \frac{b}{b} \cdot \frac{1}{c} \cdot \frac{1}{d^2};$$

the term $\frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}$ is obtained from 2 products

$$\left(\frac{b}{1}\right)\left(\frac{a}{1} \cdot \frac{b}{b} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right),$$
$$\left(\frac{a}{1}\right)\left(\frac{a}{a} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right);$$

thus the coefficient of $(1^2 \bar{1} \bar{2})$ in the product $(1)(10\bar{1}\bar{2})$ is 2, and the corresponding distributions are

$$\begin{array}{ll} A & BB \\ b & ab \\ & BBBB \\ & bcdd \end{array}, \qquad \begin{array}{ll} A & BB \\ a & ab \\ & BBBB \\ & acdd \end{array};$$

which are distributions of objects $(1^2 \bar{1} \bar{2})$ into parcels $(1 \bar{2})$, subject to the composite restriction $(1)(10\bar{1}\bar{2})$.

89. Finally we have the product

$$(10)(1\bar{1}\bar{2}) = \sum \frac{a}{1} \cdot \frac{b}{b} \sum \frac{a}{1} \cdot \frac{1}{b} \cdot \frac{1}{c^2};$$

the term $\frac{a}{1} \cdot \frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}$ is obtained from 6 products

$$\left(\frac{a}{1} \cdot \frac{b}{b}\right)\left(\frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right), \quad \left(\frac{a}{a} \cdot \frac{b}{1}\right)\left(\frac{a}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right),$$
$$\left(\frac{a}{1} \cdot \frac{c}{c}\right)\left(\frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right), \quad \left(\frac{b}{1} \cdot \frac{c}{c}\right)\left(\frac{a}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right),$$
$$\left(\frac{a}{1} \cdot \frac{d}{d}\right)\left(\frac{b}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right), \quad \left(\frac{b}{1} \cdot \frac{d}{d}\right)\left(\frac{a}{1} \cdot \frac{1}{c} \cdot \frac{1}{d^2}\right);$$

the coefficient of $(1^2\bar{1}\bar{2})$ in the product $(10)(1\bar{1}\bar{2})$ is thus 6, and the corresponding distributions are

$$\begin{array}{ll} AA & B \\ ab & b \\ A & BBB \\ b & cdd \end{array}, \quad \begin{array}{ll} AA & B \\ ab & a \\ A & BBB \\ a & cdd \end{array},$$

$$\begin{array}{ll} AA & B \\ ac & b \\ A & BBB \\ c & cdd \end{array}, \quad \begin{array}{ll} AA & B \\ bc & a \\ A & BBB \\ c & cdd \end{array},$$

$$\begin{array}{ll} AA & B \\ ad & b \\ A & BBB \\ d & cdd \end{array}, \quad \begin{array}{ll} AA & B \\ bd & a \\ A & BBB \\ d & cdd \end{array};$$

these are distributions of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$, subject to the composite restriction $(10)(1\bar{1}\bar{2})$.

90. Altogether, in the product X_1X_{-2}, the coefficient of $(1^2\bar{1}\bar{2})\,x_1^2x_0x_{-1}x_{-2}$ is 12 ($= 3 + 1 + 2 + 6$); the 12 corresponding distributions have been exhibited; each of these had reference to objects $(1^2\bar{1}\bar{2})$ and parcels $(1\bar{2})$; each, further, was associated with a composite restriction which was denoted by a separation of the partition $(1^20\bar{1}\bar{2})$ because the term $x_1^20x_{-1}x_{-2}$ has this partition; each of these separations had the specification $(1\bar{2})$ because $(1\bar{2})$ is the partition of the term X_1X_{-2}; the 12 distributions were complete, that is, they included all those that

were possible under the given conditions; this must be so because there is a one-to-one correspondence between the distributions and term products, and care was taken to consider the whole of the latter. Amongst the separations which denoted composite restrictions were included all separations of $(1^2 0 \bar{1} \bar{2})$ which had the specification $(1\bar{2})$; this is a consequence of the forms of the functions X and X_{-2}. Hence if we consider the whole cofactor of $x_1^2 x_0 x_{-1} x_{-2}$, which arises from the product $X_1 X_{-2}$ and therein the coefficient of $(1^2 \bar{1} \bar{2})$, we find that this coefficient denotes the number of ways of distributing objects $(1^2 \bar{1} \bar{2})$ into parcels $(1\bar{2})$ subject to the restriction whose partition is $(1^2 0 \bar{1} \bar{2})$; this restriction does and must involve all the composite restrictions whose separations have a specification $(1\bar{2})$, and there is no need to specifically mention the circumstance in describing the distribution; we may simply state that the analytical result

$$X_1 X_{-2} = \ldots . + 12 (1^2 \bar{1} \bar{2})\, x_1^2 x_0 x_{-1} x_{-2} + \ldots .$$

is the analytical statement of the arithmetical theorem: "There are 12 ways of distributing objects $(1^2 \bar{1} \bar{2})$ into parcels $(1\bar{2})$, subject to the restriction whose partition is $(1^2 0 \bar{1} \bar{2})$."

91. In the case just considered there is a one-to-one correspondence between the literal products and the distributions; this, however, does not always obtain. Suppose that we take the product of symmetric functions $(1^2 0)(2)$, in which each factor is of the same weight 2, and seek the coefficient of (21^2) in its development; proceeding in the usual manner, we have

$$(1^2 0)(2) = \sum \frac{a}{1} \cdot \frac{b}{1} \cdot \frac{c}{c} \sum \frac{a^2}{1}$$

and

$$(21^2) = \sum \frac{a^2}{1} \cdot \frac{b}{1} \cdot \frac{c}{1};$$

the term $\frac{a^2}{1} \cdot \frac{b}{1} \cdot \frac{c}{1}$ arises only from the product

$$\left(\frac{b}{1} \cdot \frac{c}{1} \cdot \frac{a}{a}\right)\left(\frac{a^2}{1}\right),$$

but corresponding to this decomposition, there are two distributions of 6 objects (21^2) into 6 parcels (2^2), viz.

$$\begin{array}{llcll} AAA & BB & & AA & BBB \\ abc & aa & , & aa & abc \\ A & & & & B \\ a & & & & a \end{array} ;$$

the fact is that the component partitions $(1^2 0)$ and (2) being of the *same* weight but *different*, we obtain an additional distribution by the interchange of A and B; but if we form the product X_2^2 we obtain a term $2(1^2 0)(2)$, the 2 appearing for the very reason that $(1^2 0)$ and (2) are of the same weight but different; we may therefore effect a one-to-one correspondence between the literal products in $2(1^2 0)(2)$ and the distributions thence arising. Similarly, if we form the product X_λ^l, and $(\urcorner_1)$, $\urcorner_2)$, ... denote different partitions of weight λ, we will on development obtain a term which involves

$$\frac{(l_1+l_2+\dots)!}{l_1!\,l_2!\dots}(\urcorner_1)^{l_1}(\urcorner_2)^{l_2}\dots;$$

and, moreover, corresponding to a literal product in $(\urcorner_1)^{l_1}(\urcorner_2)^{l_2}\dots$ there will be precisely

$$\frac{(l_1+l_2+\dots)!}{l_1!\,l_2!\dots}$$

distributions, since we may permute the capital letters in any one distribution in all possible ways; thus we may consider that there exists a one-to-one correspondence between the literal products in

$$\frac{(l_1+l_2+\dots)!}{l_1!\,l_2!\dots}(\urcorner_1)^{l_1}(\urcorner_2)^{l_2}\dots$$

and the distributions which arise from them.

92. In general, if partitions of the same weight p_s (where p_s is positive, zero, or negative) be denoted by (P_s'), (P_s''), (P_s'''), ..., the development of the product

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\dots$$

will produce a term involving

$$\frac{(\pi_1'+\pi_1''+\dots)!}{\pi_1'!\,\pi_1''!\dots}\cdot\frac{(\pi_2'+\pi_2''+\dots)!}{\pi_2'!\,\pi_2''!\dots}\dots(P_1)^{\pi_1'}(P_1')^{\pi_1''}\dots(P_2)^{\pi_2'}(P_2')^{\pi_2''}\dots,$$

and there will be a one-to-one correspondence between the literal terms occurring therein and the distributions arising therefrom.

93. Hence, from what has gone before, the result

$$X_{p_1}^{\pi_1}X_{p_2}^{\pi_2}X_{p_3}^{\pi_3}\dots = \dots + \theta(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots)\,x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\dots + \dots$$

is the analytical statement of the arithmetical theorem: "There are θ ways of distributing objects $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots)$ into parcels $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots)$, subject to the restriction whose partition is $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\dots)$."

94. Recalling our former result

$$X_1X_{-2} = \ldots . + 12\,(1^2\bar{1}\bar{2})\,x_1^2x_0x_{-1}x_{-2} + \ldots . ,$$

we can now establish, in an instantaneous manner, the reciprocal result

$$X_1^2X_{-1}X_{-2} = \ldots . + 12\,(1\bar{2})\,x_1^2x_0x_{-1}x_{-2} + \ldots . ;$$

for, take any one of the foregoing 12 distributions, viz.

$$\begin{array}{ll} AA & B \\ ab & a \\ A & BBB \\ a & cdd \end{array},$$

and change the small letters into capitals and vice versa, we get thus a distribution

$$\begin{array}{ll} aa & b \\ AB & A \\ a & bbb \\ A & CDD \end{array},$$

which may be put into the form

$$\begin{array}{llll} AA & B & & \\ ab & a & & \\ A & & C & DD \\ a & & b & bb \end{array},$$

and this denotes a distribution of objects $(1\bar{2})$ into parcels $(1^2\bar{1}\bar{2})$, subject to a restriction whose partition is $(1^20\bar{1}\bar{2})$.

95. We have thus passed from a distribution of objects $(1^2\bar{1}\bar{2})$ into parcels $(1\bar{2})$ to a distribution of objects $(1\bar{2})$ into parcels $(1^2\bar{1}\bar{2})$ without altering the restriction which still possesses the partition $(1^20\bar{1}\bar{2})$.

96. This interchange of small and capital letters (in reality an interchange of objects and parcels) cannot possibly alter the partition of restriction; this is manifest from the definition of the latter.

97. Further, the process is reversible; from every distribution of the second kind we are able to pass to a distribution of the first kind and vice versa.

98. There is thus a one-to-one correspondence between the two natures of distribution, and the numbers of the distributions of the two kinds must be identical. Hence

$$X_1^2X_{-1}X_{-2} = \ldots . + 12\,(1\bar{2})\,x_1^2x_0x_{-1}x_{-2} + \ldots . ,$$

for this is merely the analytical statement of the arithmetical fact that there are 12 distributions of objects $(1\bar{2})$ into parcels $(1^2\bar{1}\bar{2})$ subject to a restriction whose partition is $(1^2 0\bar{1}\bar{2})$.

99. The general theorem is now practically established, for if

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \dots = \dots + \theta\,(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots)\, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \dots + \dots,$$

there are θ ways of distributing objects $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots)$ into parcels $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots)$ subject to a restriction whose partition is $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\dots)$, and the above reversible process proves that there must be also exactly θ ways of distributing objects $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots)$ into parcels $(\lambda_1^{l_1}\lambda_2^{l_2}\lambda_3^{l_3}\dots)$, subject to the same restriction; hence

$$X_{\lambda_1}^{l_1} X_{\lambda_2}^{l_2} X_{\lambda_3}^{l_3} \dots = \dots + \theta\,(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\dots)\, x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \dots + \dots,$$

the theorem to be demonstrated.

100. This proposition is cardinal in symmetrical algebra and of great importance; I hope, in a subsequent memoir in this Journal, to give another proof of it by means of differential operators.

SECTION 6.

The Formation of Symmetrical Tables.

101. One of the consequences of the theorem of reciprocity is the possibility of forming a pair of tables of symmetric functions, of a symmetrical character, in association with every partition, in positive, zero, and negative integers, of every number, positive, zero, or negative.

102. For, let the separations of the partition $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\dots)$ possess in all r specifications which may be

$$\varkappa_1,\ \varkappa_2,\ \varkappa_3,\ \dots\ \varkappa_r,$$

and let, moreover,

$$[X_{\kappa_1}],\ [X_{\kappa_2}],\ [X_{\kappa_3}],\ \dots\ [X_{\kappa_r}]$$

denote the corresponding X-products, so that if

$$\varkappa_m = (\mu_1\mu_2\mu_3\dots),$$
$$[X_{\kappa_m}] = X_{\mu_1}X_{\mu_2}X_{\mu_3}\dots.$$

103. The law of reciprocity shows that if

$$X_{p_1}^{\pi_1} X_{p_2}^{\pi_2} X_{p_3}^{\pi_3} \dots = \dots + P x_{s_1}^{\sigma_1} x_{s_2}^{\sigma_2} x_{s_3}^{\sigma_3} \dots + \dots,$$

so that P consists of an assemblage of separations of the partition $(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots)$, each of which has the specification $(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)$ which is one of the series of specifications

$$\varkappa_1,\ \varkappa_2,\ \varkappa_3,\ \ldots\ldots\ \varkappa_r,$$

P on development will only give rise to symmetric functions which are symbolized by partitions included in the specification set $\varkappa_1,\ \varkappa_2,\ \varkappa_3,\ \ldots\ldots\ \varkappa_r$; for otherwise the law of reciprocity could not be true.

104. Now form X-products corresponding to all the specifications; let $P_{\kappa_1},\ P_{\kappa_2},\ P_{\kappa_3},\ \ldots\ldots\ P_{\kappa_r}$ be the corresponding values of P, and further let $\theta_{t,\,m}$ be the numerical coefficient of $\varkappa_m x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots$ in the development of $[X_{\kappa_t}]$, or what is the same thing, the coefficient of $\varkappa_m$ in the development of P_{κ_t}; thus,

$$\begin{aligned}
X_{\kappa_1} &= \ldots\ldots + P_{\kappa_1}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots + \ldots\ldots \\
&= \ldots\ldots + (\theta_{1,\,1}\varkappa_1 + \theta_{1,\,2}\varkappa_2 + \theta_{1,\,3}\varkappa_3 + \ldots\ldots + \theta_{1,\,r}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots + \ldots\ldots, \\
X_{\kappa_2} &= \ldots\ldots + P_{\kappa_2}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots + \ldots\ldots \\
&= \ldots\ldots + (\theta_{21}\varkappa_1 + \theta_{22}\varkappa_3 + \theta_{23}\varkappa_3 + \ldots\ldots + \theta_{2r}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots + \ldots\ldots, \\
X_{\kappa_3} &= \ldots\ldots + P_{\kappa_3}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots + \ldots\ldots \\
&= \ldots\ldots + (\theta_{31}\varkappa_1 + \theta_{32}\varkappa_2 + \theta_{33}\varkappa_3 + \ldots\ldots + \theta_{3r}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots + \ldots\ldots, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
X_{\kappa_r} &= \ldots\ldots + P_{\kappa_r}x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_3}^{\sigma_3}\ldots\ldots + \ldots\ldots \\
&= \ldots\ldots + (\theta_{r1}\varkappa_1 + \theta_{r2}\varkappa_2 + \theta_{r3}\varkappa_3 + \ldots\ldots + \theta_{rr}\varkappa_r)\, x_{s_1}^{\sigma_1}x_{s_2}^{\sigma_2}x_{s_2}^{\sigma_3}\ldots\ldots + \ldots\ldots
\end{aligned}$$

105. This result shows that the assemblages of separations

$$P_{\kappa_1},\ P_{\kappa_2},\ P_{\kappa_3},\ \ldots\ldots\ P_{\kappa_r}$$

are linearly connected with the specifications

$$\varkappa_1,\ \varkappa_2,\ \varkappa_3,\ \ldots\ldots\ \varkappa_r,$$

and that we may form a table, viz.

	$\varkappa_1$	$\varkappa_\kappa$	$\varkappa_3$	....	$\varkappa_r$
P_{κ_1}	θ_{11}	θ_{12}	θ_{13}	....	θ_{1r}
P_{κ_2}	θ_{21}	θ_{22}	θ_{23}	....	θ_{2r}
P_{κ_3}	θ_{31}	θ_{32}	θ_{33}	....	θ_{3r}
	...	...	...	.....	...
P_{κ_r}	θ_{r1}	θ_{r2}	θ_{r3}	....	θ_{rr}

and then by the law of reciprocity

$$\theta_{pq} = \theta_{qp},$$

and the table enjoys row and column (i. e. diagonal) symmetry.

106. We may similarly invert the table and express $\varkappa_1, \varkappa_2, \varkappa_3, \ldots \varkappa_r$ as linear functions of $P_{\kappa_1}, P_{\kappa_2}, P_{\kappa_3}, \ldots P_{\kappa_r}$ in a table enjoying the same symmetry.

107. To make the meaning clear, omit in the first instance all partitions which contain zero or negative parts, and write down a complete system of X-products for any given weight, as follows, e. g. weight $= 5$:

	5 X_5	41 X_4X_1	32 X_3X_2	31^2 $X_3X_1^2$	2^21 $X_2^2X_1$	21^3 $X_2X_1^3$	1^5 X_1^5
x_5	(5)						
x_4x_1	(41)	$(4)(1)$					
x_3x_2	(32)		$(3)(2)$				
$x_3x_1^2$	(31^2)	$(31)(1)$	$(3)(1^2)$	$(3)(1)^2$			
$x_2^2x_1$	(2^21)	$(2^2)(1)$	$(2)(21)$		$(2)^2(1)$		
$x_2x_1^3$	(21^3)	$(21^2)(1)$	$(2)(1^3)+(21)(1^2)$	$(21)(1)^2$	$2\,(2)(1^2)(1)$	$(2)(1)^3$	
x_1^5	(1^5)	$(1^4)(1)$	$(1^3)(1^2)$	$(1^3)(1)^2$	$(1^2)^2(1)$	$(1^2)(1)^3$	$(1)^5$

here each line is a set of "assemblages of separations," each assemblage having its own specification, as appearing by the top line. The assemblages and specifications represent symmetric functions, and the theorem is that these symmetric functions are linearly connected, the coefficients being symmetrical in regard to a diagonal. Thus, from the last line but one we have the assemblages (separations of (21^3))

$$(21^3),\ (21^2)(1),\ (2)(1^3)+(21)(1^2),\ (21)(1)^2,\ 2\,(2)(1^2)(1),\ (2)(1)^3$$

linearly connected with the specifications

$$(5),\quad (41),\quad (32),\quad (31^2),\quad (2^21),\quad (21^3).$$

108. Again, let us take the weight -2 and the separable partition $(0^2\bar{1}^2)$; the corresponding portion of the table of X-products is

	$(\bar{2})$ X_{-2}	$(0\bar{2})$ X_0X_{-2}	$(0^2\bar{2})$ $X_0^2X_{-2}$	$(\bar{1}^2)$ X_{-1}^2	$(0\bar{1}^2)$ $X_0X_{-1}^2$	$(0^2\bar{1}^2)$ $X_0^2X_{-1}^2$
$x_0^2x_{-1}^2$	$(0^2\bar{1}^2)$	$(0)(0\bar{1}^2)+(0^2)(\bar{1}^2)$	$(0)^2(\bar{1}^2)$	$2(0^2\bar{1})(\bar{1})+(0\bar{1})^2$	$(0^2)(\bar{1})^2+2(0)(0\bar{1})(\bar{1})$	$(0)^2(\bar{1})^2$

showing that we have the assemblages (separations of $(0^2\bar{1}^2)$) indicated in the bottom line, linearly connected with the specifications shown in the top line.

109. Writing down the assemblages in a vertical column and the specifications in a horizontal row, we may then form a table which calculation shows to be,

	$(\bar{2})$	$(0\bar{2})$	$(0^2\bar{2})$	$(\bar{1}^2)$	$(0\bar{1}^2)$	$(0^2\bar{1}^2)$
$(0^2\bar{1}^2)$						1
$(0)(0\bar{1}^2)+(0^2)(\bar{1}^2)$				1	5	3
$(0)^2(\bar{1}^2)$				4	5	2
$2(0^2\bar{1})(\bar{1})+(0\bar{1})^2$		1	4	2	10	8
$(0^2)(\bar{1})^2+2(0)(0\bar{1})(\bar{1})$		5	5	10	20	10
$(0)^2(\bar{1})^2$	1	3	2	8	10	4

where the third line is to be read

$$(0)^2(\bar{1}^2) = 4(\bar{1}^2) + 5(0\bar{1}^2) + 2(0^2\bar{1}^2),$$

or in an algebraic form

$$\left(\sum \alpha^0\right)^2 \sum \alpha^{-1}\beta^{-1} = 4\sum \alpha^{-1}\beta^{-1} + 5\sum \alpha^0\beta^{-1}\gamma^{-1} + 2\sum \alpha^0\beta^0\gamma^{-1}\delta^{-1},$$

verified through the medium of the identity

$$n^2 \qquad = 4 \qquad + 5(n-2) \qquad + 2.\tfrac{1}{2}(n-2)(n-3).$$

110. The table already given possesses diagonal symmetry as a direct consequence of the law of reciprocity; the inverse table, which expresses the specifi-

cations as linear functions of the assemblages of separations, necessarily enjoys the same symmetry. Its form is

	$(0^2\bar{1}^2)$	$(0)(0\bar{1}^2)+(0^2)(\bar{1}^2)$	$(0)^2(\bar{1}^2)$	$2(0^2\bar{1})(\bar{1})+(0\bar{1})^2$	$(0^2)(\bar{1})^2+2(0)(0\bar{1})(\bar{1})$	$(0)^2(\bar{1})^2$
$(\bar{2})$	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{2}{3}$	1
$(0\bar{2})$	$\frac{2}{3}$	$\frac{2}{15}$	$-\frac{8}{15}$	$-\frac{1}{3}$	$\frac{4}{15}$	
$(0^2\bar{2})$	$-\frac{2}{3}$	$-\frac{8}{15}$	$\frac{2}{15}$	$\frac{1}{3}$	$-\frac{1}{15}$	
$(\bar{1}^2)$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$			
$(0\bar{1}^2)$	$-\frac{2}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$			
$(0^2\bar{1}^2)$	1					

111. It has thus been demonstrated that a pair of symmetrical tables exist in the case of every partition into positive, zero, and negative integers of every number positive, zero, or negative.

112. The theorem in regard to the coefficients in a group, given on page 35 of the former memoir, is extended easily to this enlarged theory, and we may enunciate as follows:

113. *Theorem.* "In the expression of symmetric function

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)$$

by means of separations of

$$(s_1^{\sigma_1}s_2^{\sigma_2}s_3^{\sigma_3}\ldots\ldots),$$

where the parts of the partitions are positive, zero, or negative, the algebraic sum of the coefficients in each group will be zero if the partition

$$(p_1^{\pi_1}p_2^{\pi_2}p_3^{\pi_3}\ldots\ldots)$$

possesses no separation of specification

$$(\sigma_1 s_1,\ s_2\sigma_2,\ s_3\sigma_3,\ \ldots\ldots).\text{''}$$

114. This theorem may be verified in the second of the tables above given in the cases of $(\bar{2})$ and $(\bar{1}^2)$ only, as all the other symmetric functions in the left-hand vertical column possess separations of specification $(0\bar{2})$. Ex. gr.

$$\begin{aligned}(\bar{2}) = {} & (0)^2(\bar{1})^2 - \tfrac{2}{3}(0^2)(\bar{1})^2 - \tfrac{2}{3}(0)^2(\bar{1}^2) + \tfrac{2}{3}(0^2)(\bar{1}^2) \\ & - \tfrac{4}{3}(0)(0\bar{1})(\bar{1}) + \tfrac{2}{3}(0^2\bar{1})(\bar{1}) + \tfrac{2}{3}(0)(0\bar{1}^2) - \tfrac{2}{3}(0^2\bar{1}^2) \\ & + \tfrac{1}{3}(0\bar{1})^2\end{aligned}$$

$$\pm\tfrac{4}{3} \qquad \pm\tfrac{2}{3} \qquad \pm\tfrac{2}{3} \qquad \pm\tfrac{2}{3}$$

$$\begin{aligned}(\bar{1}^2) = {} & \tfrac{1}{3}(0)^2(\bar{1}^2) - \tfrac{1}{3}(0^2)(\bar{1}^2) \\ & - \tfrac{1}{3}(0)(0\bar{1}^2) + \tfrac{1}{3}(0^2\bar{1}^2)\end{aligned}$$

$$\pm\tfrac{1}{3} \qquad \pm\tfrac{1}{3}.$$

SECTION 7.

The Law of Expressibility.

115. The law given on page 6 of former memoir may now be extended as follows:

116. *Theorem.* "If a symmetric function be symbolized by

$$(\lambda\mu\nu\ \ldots\ldots)$$

the parts λ, μ, ν, ... being positive, zero, or negative, and

$(\lambda_1\lambda_2\lambda_3\ \ldots\ldots)$ be any partition of λ,
$(\mu_1\mu_2\mu_3\ \ldots\ldots)$ " " " μ,
$(\nu_1\nu_2\nu_3\ \ldots\ldots)$ " " " ν,
. . . .

the symmetric function $(\lambda\mu\nu\ \ldots\ldots)$

is expressible as a linear function of separations of

$$\lambda_1\lambda_2\lambda_3\ \ldots\ldots\ \mu_1\mu_2\mu_3\ \ldots\ \nu_1\nu_2\nu_3\ \ldots\ldots).\text{''}$$

117. As an example of this, we may express the function (0^2) as a linear

function of separations of (0^4); it will be interesting to give, as well, the complete tables of separations of (0^4) which includes this result.

	(0)	(0^2)	(0^3)	(0^4)
$(0)^4$	1	14	36	24
$3(0^2)(0)^2$		12	45	36
$(0^2)^2 + 2(0^3)(0)$		1	12	14
(0^4)				1

	$(0)^4$	$3(0^2)(0)^2$	$(0^2)^2 + 2(0^3)(0)$	(0^4)
(0)	1	$-\frac{4}{3}$	2	-4
(0^2)		$\frac{4}{33}$	$-\frac{5}{11}$	2
(0^3)		$-\frac{1}{99}$	$\frac{4}{33}$	$-\frac{4}{3}$
(0^4)				1

from which

$$(0^2) = \tfrac{4}{33} \cdot 3(0^2)(0)^2 - \tfrac{5}{11}\{(0^2)^2 + 2(0^3)(0)\} + 2(0^4),$$

and this merely exhibits a relation connecting the second, third, fourth and fifth binomial coefficients in the expansion of $(1 + x)^n$; for

$$(1 + x)^n = 1 + (0)\,x + (0^2)\,x^2 + (0^3)\,x^3 + (0^4)\,x^4 + \ldots . + (0^n)\,x^n.$$

118. The subject of "Expansion by factorials," which is usually discussed in works on Finite Differences, is thus clearly within the domain of this theory, and the two tables last given might have been expressed by the notation and symbols of the calculus of Finite Differences.

On this subject I hope to say more upon a future occasion.

119. It is, in conclusion, to be particularly observed that all algebra is expressible by means of factorials, and thus any algebraical expression whatever, of a finite nature, may be exhibited as a symmetric function of one or more sets of quantities.

Royal Arsenal, Woolwich, England, December 1st, 1888.